Design of Branched and Unbranched Axially Symmetrical Ducts with Specified Pressure Distribution

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This paper concerns the potential flow design of axially symmetrical ducts of both circular and annular cross section with or without wall suction or blowing slots. The objective of the work was to develop a method by which such ducts could be designed with directly prescribed wall pressure variation. Previous axially symmetrical design methods applied only to circular cross-sectional ducts and required that the pressure distribution be prescribed along the duct centerline and not along the duct wall. The present method uses an inverse problem approach which extends the method of Stanitz to the axially symmetrical case, and an approximation is used to account for the stagnation point in branched duct designs. Two examples of successful designs of diffusers with suction slots are presented.

Nomenclature

n	= coordinate along equipotential lines in natural coordinate system
R	= radial axially symmetrical coordinate
R_0	= first approximation to the radial coordinate function describing the duct flowfield
s	= coordinate along streamlines in natural coordinate system
U	= velocity
X	= axial axially symmetrical coordinate
α	= angle between velocity vector and positive X axis
ϕ	= velocity potential function
Φ	= velocity potential value at right equipotential line
ψ	= stream function
Ψ	= stream function value on wall streamline

Introduction

OFTEN in the design of fluid flow systems, one wishes to determine the shape of a flow contraction or expansion which yields a particular wall pressure variation. Inverse problem potential flow design methods for unbranched, planar ducts with prescribed wall pressure variation have existed from some time. Design methods such as those of Lighthill[1] and Stanitz[2] allow one to design essentially any type of planar duct, as long as it contains no branch points, such as occur in ducts with suction and blowing slots. If one were faced with the design of an axially symmetrical duct of either circular or annular cross section, however, there previously existed no method which allowed the direct determination of a wall shape that could meet a particular wall pressure distribution requirement. Several methods did exist for circular cross-sectional ducts which allowed the design of a duct by describing the velocity distribution along the duct centerline,[3-8] but with these methods the designer had to accept the wall pressure distribution dictated to him by the particular method. To the authors' knowledge, no exact inverse problem design method of any type for annular ducts previously existed. In the case of previously existing circular cross-sectional duct design methods, as with the two-dimensional case, branching points were not allowed.

It is the purpose of this paper to present an exact design method for axially symmetrical ducts of both circular and annular cross section which allows for the direct prescription of desired wall pressure distribution. In addition, a method of handling the problem of design of ducts with branch points, such as those encountered in the design of ducts with suction and blowing slots, also is presented.

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Design Method

The problem of axially symmetrical duct design considered here will be approached from the inverse problem standpoint. In such problems one wishes to find the geometry of a duct which will have a certain wall pressure or velocity variation, since the pressure and velocity are related by the Bernoulli equation. In order to find this geometry, the governing equations of the flow first are transformed into a coordinate system which is independent of geometry. These equations then are solved for the velocity field, and, finally, an inverse transformation yields the desired geometry.

Design Equations

The equations for the present inverse problem potential flow design method can be obtained by performing a double transformation of the governing equations for incompressible, irrotational, axially symmetrical flow. The equations of continuity and irrotationality in terms of the velocity potential

$$\frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial R^2} + \frac{1}{R}\frac{\partial \phi}{\partial R} = 0 \tag{1}$$

and

$$\frac{\partial^2 \phi}{\partial R\, \partial X} - \frac{\partial^2 \phi}{\partial X\, \partial R} = 0 \tag{2}$$

first are transformed onto the natural coordinate plane by use of a transformation defined by the equations

$$ds = \cos\alpha\, dX + \sin\alpha\, dR \tag{3}$$

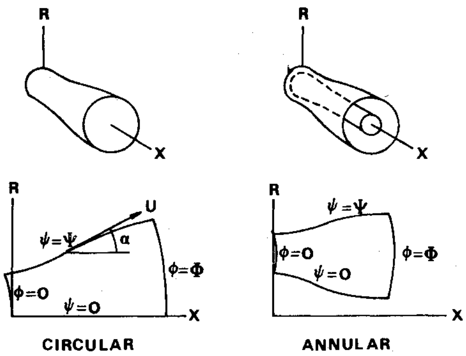

Fig. 1 Duct geometries on the physical plane.

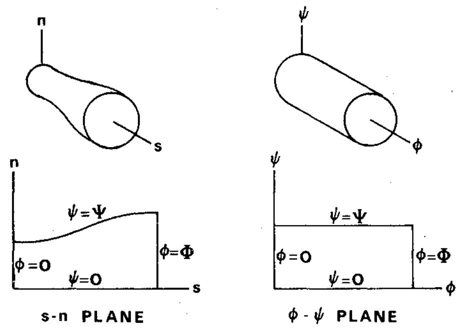

Fig. 2 Duct geometries on the natural coordinate and $\phi\psi$ planes.

and

$$dn = -\sin\alpha\, dX + \cos\alpha\, dR \tag{4}$$

The resulting equations then are transformed onto a $\phi\psi$ plane by use of a transformation defined by the definitions of the stream function and velocity potential function

$$d\psi = R\, U\, dn \tag{5}$$

and

$$d\phi = U\, ds \tag{6}$$

The transformation from the physical to $\phi\psi$ plane is illustrated by Figs. 1 and 2. After transformation, Eqs. (1) and (2) become

$$\frac{1}{R}\frac{\partial \ln U}{\partial \phi} + \frac{1}{R^2}\frac{\partial R}{\partial \phi} + \frac{\partial a}{\partial \phi} = 0 \tag{7}$$

and

$$R\frac{\partial \ln U}{\partial \psi} - \frac{\partial \alpha}{\partial \phi} = 0 \tag{8}$$

Differentiation and combination of Eqs. (7) and (8) to eliminate α produce the following governing differential equation:

$$R^4\frac{\partial^2 \ln U}{\partial \psi^2} + R^3\frac{\partial R}{\partial \psi}\frac{\partial \ln U}{\partial \psi} + R^2\frac{\partial^2 \ln U}{\partial \phi^2}$$
$$-R\frac{\partial R}{\partial \phi}\frac{\partial \ln U}{\partial \phi} + R\frac{\partial^2 R}{\partial \phi^2} - 2\left[\frac{\partial R}{\partial \phi}\right]^2 = 0 \tag{9}$$

The inverse transformation back to the physical plane can be defined by constructing the differentials of α, X, and R in the following manner:

$$d\alpha = R\frac{\partial \ln U}{\partial \psi}d\phi - \left[\frac{1}{R}\frac{\partial \ln U}{\partial \phi} + \frac{1}{R^2}\frac{\partial R}{\partial \phi}\right]d\psi \tag{10}$$

$$dX = \frac{\cos\alpha}{U}d\phi - \frac{\sin\alpha}{R\,U}d\psi \tag{11}$$

$$dR = \frac{\sin\alpha}{U}d\phi + \frac{\cos\alpha}{R\,U}d\psi \tag{12}$$

Equations (9-12) constitute a set of equations which, given suitable boundary conditions, can be solved for the desired geometry of an axially symmetrical duct. In this form the equations are very similar to those derived by Stanitz[2] for the planar case, except that, in this case, Eq. (9) contains an additional dependent variable R, and, thus, the problem must be solved on the transformed and physical planes simultaneously.

It is interesting to note that the preceding formulation of the problem is equivalent to the formulation of Jeppson,[9] who has used his problem formulation to solve analysis problems with free surfaces. The formulation of the problem described here differs from that used by Jeppson, however, in that Jeppson's method requires the solution of a nonlinear differential equation, whereas in the present method the problem is reduced to an iterative solution of a linearized differential equation as will be described in the following section.

Method of Solution

The iterative solution of Eqs. (9-12) is initiated by assuming values of the radial coordinate R for all points in the flowfield. Successive approximations to the functions $\ln U$ and R then are obtained by solving Eqs. (9, 10, and 12) in an iterative manner as described in the following.

The first step is to assume that the function R is some constant greater than zero, denoted by R_0. For circular cross-sectional ducts, R_0 is chosen as 1.0, and for annular ducts it is chosen as some constant which locates the annulus at a particular distance from the axis (the case where $R_0 = 1.0$ corresponds to locating the annulus a distance equal to the reference distance from the axis). This assumption reduces Eq. (9) to the form

$$R_0^2\frac{\partial^2 \ln U}{\partial \psi^2} + \frac{\partial^2 \ln U}{\partial \phi^2} = 0 \tag{13}$$

For the case where $R_0 = 1.0$, Eq. (13) reduces to the plane flow form developed by Stanitz.

The next step in the solution is to solve Eq. (13) using specified boundary conditions to obtain a first approximation to the function $\ln U$. This function then is used to obtain a second approximation to the function R by first intergrating Eq. (10), with $R = R_0$, along streamlines and equipotential lines to obtain an approximation to the flow angle function α and then evaluating the following line integrals:

$$R = \int_\psi \frac{\sin\alpha}{U}d\phi \tag{14}$$

$$R^2 = 2\int_\phi \frac{\cos\alpha}{U}d\psi \tag{15}$$

Equations (14) and (15) were obtained by evaluating Eq. (12) along streamlines and equipotential lines, respectively. For circular ducts, integration is begun on the axis where both R and α are zero, and for annular ducts integration is begun at some point where R and α are specified. In the present method, R was specified at the lower left-hand grid point, and α was specified on the left end of any desired streamline. At this time the first iteration through the governing equations has been completed.

The approximation to the radial coordinate function just obtained is used next to calculate values of all coefficients in Eq. (9), which then is solved for a new approximation to the function $\ell n\, U$. A new approximation to the function R then is obtained by solving Eqs. (10, 14, and 15). This process of solving sequentially Eqs. (9, 10, 14, and 15) is continued until sufficiently accurate approximations to the functions $\ell n\, U$ and R are obtained. After the iteration process is completed, the axial coordinates of all points of the flowfield are obtained by integration of Eq. (11) along streamlines and equipotential lines.

Special consideration is needed on the X axis in the design of circular cross-sectional ducts. On the X axis, where $R=0$, all of the terms in Eq. (9) go to zero, and, thus, no information can be obtained about the velocity along the center streamline. This difficulty can be circumvented by expanding $\ell n\, U$ in a Taylor series in the following manner about the center streamline:

$$\ell n\, U(\phi,\Delta\psi) = \ell n\, U(\phi,0) + \frac{\partial \ell n\, U}{\partial \psi}(\phi,0)\Delta\psi + \ldots$$

by symmetry, however,

$$\frac{\partial \ell n\, U}{\partial \psi}(\phi,0) = 0$$

Thus, to second-order accuracy

$$\ell n\, U(\phi,\Delta\psi) \simeq \ell n\, U(\phi,0)$$

which states that the velocity on the center streamline can be approximated to second-order accuracy by the velocity computed a small distance from the axis.

At first glance, it might appear that difficulties also would be encountered when integrating the second terms of Eqs. (10) and (11) on the X axis. Upon closer inspection of these terms, however, it can be seen that they are actually $\partial\alpha/\partial\psi$ and $\partial X/\partial\psi$, respectively, which are zero on the axis because of symmetry.

Boundary Conditions

Two types of boundary conditions are needed for solution of Eq. (9) on the transformed plane. These are the boundary conditions on the wall streamlines and on the inlet and outlet equipotential lines.

Boundary conditions for the velocity on the wall streamlines are prescribed in terms of the arc length along the wall streamline. These boundary conditions then are transformed in order to obtain the velocity as a function of ϕ by using the transformation

$$\phi = \int_{\psi} U \, ds$$

Two types of boundary conditions on the inlet and outlet equipotential lines produce physically significant results. The magnitude of the velocity (Dirichlet boundary condition) or the flow direction (Neumann boundary condition) can be prescribed on the inlet and outlet equipotential lines. If the flow direction is to be prescribed, both $\partial \ell n\, U/\partial\phi$ and $\partial R/\partial\phi$ must be prescribed along the equipotential line in question, as can be seen from an examination of Eq. (10). If both derivatives are set equal to zero, a parallel flow results.

The design of ducts with suction or blowing slots requires more complicated boundary conditions along the duct wall. The new boundaries along the duct wall for the design of branched ducts are illustrated in Fig. 3. Boundary conditions now must be supplied along streamlines A, C, and D, and equipotential line B, where previously they were supplied only along a single streamline. On streamlines A, C, and D, the velocity is prescribed as a boundary condition, whereas along the equipotential line B, parallel flow boundary conditions are prescribed.

In the design of branched ducts there are three possible configurations for the branching point: cusp, wedge, or blunt. A cusped branch point was not used in the present case because of the difficulty in manufacturing the sharp edges produced by such a design. The blunt branch point also was discarded since the method was to be used to design diffusers with relatively small suction slots, and it was not possible to manufacture accurately such small slot lips of complicated geometry. Thus, a simple wedge was chosen for the branch point geometry. The use of a wedge-shaped branch point involves the problem of handling a stagnation point on the boundary as shown in Figm 3. The presence of a stagnation point on the boundary introduces a singularity into Eq. (9), since $\ell n\, U$ approaches minus infinity as U approaches zero. In order to deal with this difficulty, an approximation was introduced into the flowfield in the vicinity of the stagnation point. This approximation consists of substituting into the flowfield the solution for flow around a two-dimensional wedge. Since the wedge flow solution is a planar solution, results computed for the axially symmetrical case by this approximation would not be expected to be good near the axis where the radial effect is large. However, good accuracy should be achieved some distance from the axis where the slot is located. A more complete discussion of the wedge flow approximation is presented in Ref. 10.

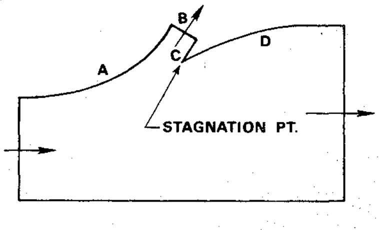

Fig. 3 Geometry of branched duct.

Computer Solutions

The preceding method was programmed for an IBM 360 computer using finite-difference techniques with three-point

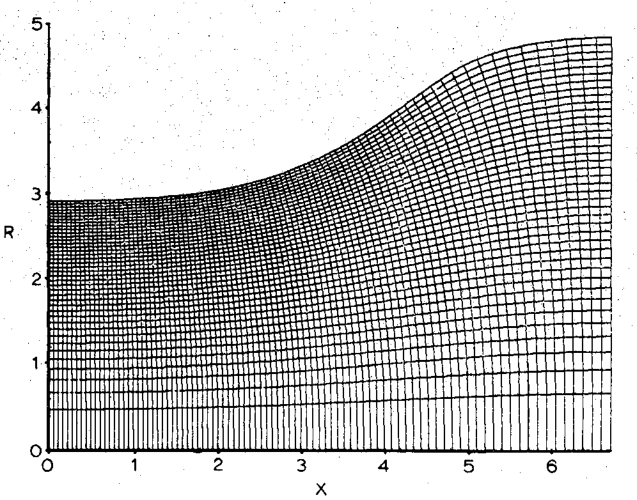

Fig. 4 Flowfield in circular cross-sectional duct with parallel flow at inlet and outlet.

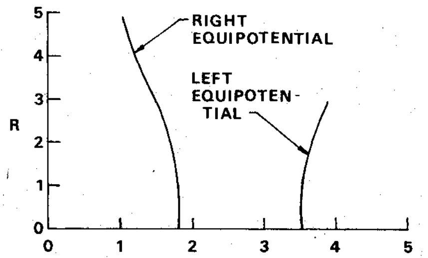

Fig. 5 Velocity profiles at inlet and outlet of circular cross-sectional duct.

centered differences at interior points and three-point forward and backward differences on the boundaries. A Gauss-Seidel method was used to solve the resulting set of equations, and Simpson's integral method was used for calculating all integrals. Two computer programs have been developed: one for general use in the design of axially symmetrical ducts[11] and one for design of ducts with suction or blowing slots.[10] The solution technique used in the program yields results with accuracy of the order of the spacing to the fourth power for an equally spaced grid and of the order of the spacing to the third power for unequal spacing, except on the axis for circular cross-sectional ducts as was noted previously. Several examples of ducts designed by the method are presented below. In all of the cases, convergence on the radial coordinate with accuracy better than 0.05% was achieved in 10 iterations or less on the radial coordinate with 50 iterations per radial iteration on $\ln U$. For all cases R_0 was assumed to be a constant throughout the field.

Results for Unbranched Ducts

Two examples of the use of the method for the design of unbranched axially symmetrical ducts have been included to illustrate the capabilities of the method. The first example, shown in Figs. 4 and 5, illustrates the use of the method for design of a circular cross-sectional duct for which parallel flow boundary conditions have been used at the inlet and outlet of the duct. The streamlines and equipotential lines for the flow in this duct are shown on Fig. 4, and it can be seen that all of the streamlines have zero slope at the inlet and outlet. An equal spacing of 0.333 for streamlines and equipotential lines was used in the design, and, thus, the same quantity of flow would pass between any two adjacent streamlines. The importance of the radial coordinate can be seen readily by noting the decrease in spacing between successive streamlines away from the axis. The wall length of this duct is 3.6 units. The velocity prescribed along the wall was held at 4 for the first 0.8 units, at 1 for the last 0.8 units, and between an "S"-shaped cubic variation was used.

The prescription of parallel flow at the inlet and outlet of a duct, for finite length ducts, generally will result in a nonuniformity of inlet and outlet velocity profiles. Velocity profiles at the inlet and outlet of the duct in Fig. 4 are shown plotted against radius in Fig. 5. The figure shows that the velocity at the left end of the duct is higher at the wall than at the centerline, and that the reverse is true at the right end of the duct. Both conditions are a direct result of the prescription of parallel flow at these planes.

A second example illustrates the use of the method for design of an annular cross-sectional duct for which a uniform velocity has been prescribed on the inlet and outlet equipotential lines. The streamlines and equipotential lines within the duct are shown on Fig. 6. The geometry is shown on a shifted set of axes for improved resolution, and the centerline is actually 2.0 units below the axis shown in the figure. An equal spacing of 0.1 for streamlines and equipotential lines was used in the design. The total wall length of both walls was 4.5 units, and the same velocity distribution was prescribed for both walls. The velocity was held at 4 for the first unit of wall length, at 1 for the last 2.5 units, and between an "S"-shaped cubic was used. The radius and flow angle were specified at the lower left-hand grid point. Although the prescription of a uniform velocity on the left end of the duct did not result in any perceptible nonparallelism of the flow on this plane, a substantial nonparallelism of the flow resulted from the same requirement at the right bounding equipotential line. It is most interesting to note also that even though the same velocity distribution was prescribed on both duct walls the resulting design shows a marked asymmetry in the shapes of these two wall streamlines. This asymmetry is due to the presence of the radial coordinate in the governing equations and is an indirect indication of the relative importance of the radial coordinate in the equations.

Results for Branched Ducts

The computer program described in Ref. 10 was used in the design of the diffusers with suction slots described in Ref. 12. The geometry of the circular cross-sectional diffuser is shown in Fig. 7. Flow through the diffuser is from left to right, and a nozzle section was included as part of the design upstream of the diffuser. A grid spacing of 0.15 was used over most of the

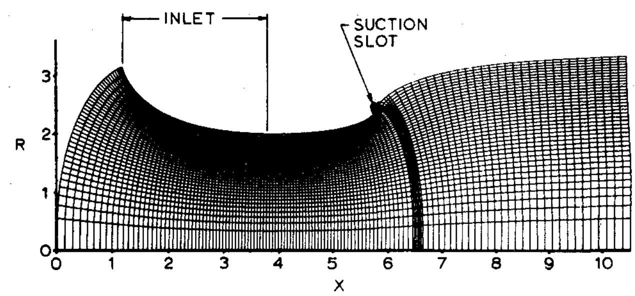

Fig. 7 Flowfield in circular cross-sectional diffuser with a suction slot.

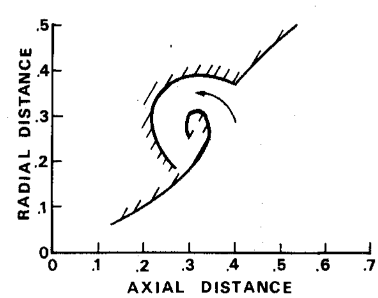

Fig. 8 Geometry of suction slot in circular cross-sectional diffuser.

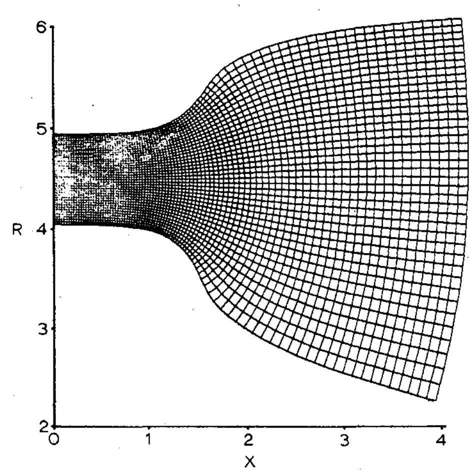

Fig. 6 Flowfield in annular cross-sectional duct with uniform velocity at inlet and outlet.

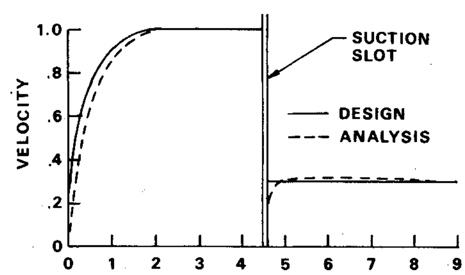

Fig. 9 Comparison of design and calculated wall velocity in circular cross-sectional diffuser.

diffuser, and a grid spacing of 0.05 was used in the area of the suction slot for improved resolution. The velocity along the wall in the nozzle was raised quadratically from 1.0 to 3.2 and was held constant from the diffuser inlet into the slot. The wall velocity downstream of the slot was held constant at 0.95. An enlarged drawing of the suction slot is shown in Fig. 8.

In order to check the accuracy of the branched duct approximation used in the design, the diffuser was analyzed using the Douglas-Neumann technique,[13] and a comparison of wall velocity is shown on Fig. 9. The discrepancy between prescribed and calculated wall velocity in the inlet nozzle is caused by the presence of a stagnation point near the inlet in the Douglas-Neumann calculation. The rounding of the velocity curve immediately downstream of the slot can be attributed to the inaccuracy introduced by the wedge approximation at the stagnation point.

The geometry computed by the design method for the 3:1 area ratio annular diffuser described in Ref. 12 is shown in Fig. 10. The geometry of this duct is shown on shifted axes, and the centerline is actually 1 unit below the horizontal axis. The flow through the diffuser would be from left to right and, as with the circular cross-sectional diffuser, a nozzle portion has been included as part of the design. Suction slots were located on both the inner and outer walls of the diffuser. A grid spacing of 0.1 was used in the streamwise direction except in the area of the slots, where it was changed to 0.025 for better resolution. In the crossstream direction, the spacing was 0.3 over the main part of the flow, 0.15 near the dividing streamline, and was varied between 0.3 and 0.025 in the vicinity of and within the slots. The same velocity distribution was prescribed on both walls. The velocity in the nozzle was raised quadratically from 1.0 to 3.0 and was held constant into the slots. The wall velocity downstream of the slots was held constant at 0.933. A radius of 1.7 was specified at the lower left grid point, and a flow angle of zero was specified on the left end of the streamline which is midway through the field. An enlarged drawing of both suction slots is shown on Fig. 11. As was the case with the annular duct described earlier, an asymmetry between upper and lower walls again was induced by radial effects, even though the same velocity distribution was prescribed on both walls.

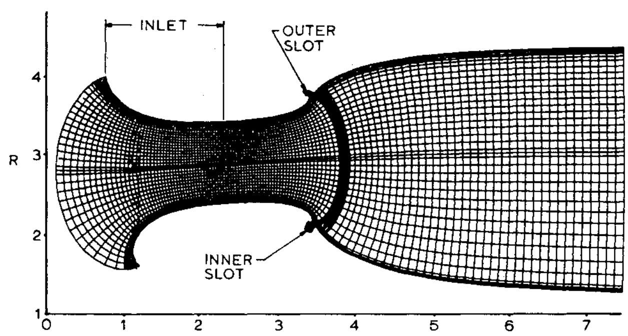

Fig. 10 Flowfield in annular cross-sectional diffuser with suction slots.

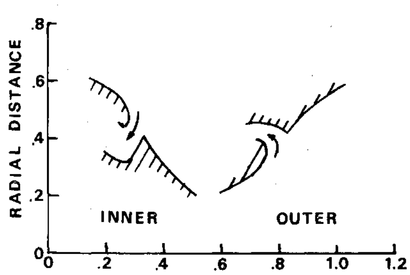

Fig. 11 Geometries of inner and outer suction slots in annular cross-sectional diffuser.

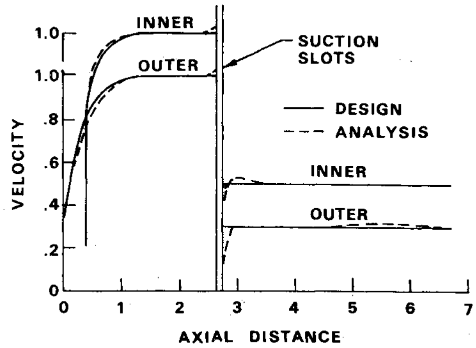

Fig. 12 Comparison of design and calculated wall velocity in annular cross-sectional diffuser.

This diffuser also was analyzed by the Douglas-Neumann technique, and prescribed and calculated wall velocities are compared on Fig. 12. The velocity variations on both the inner and outer walls of the diffuser are shown plotted on shifted axes for resolution. As with the circular cross-sectional diffuser, small discrepancies were induced in the inlet nozzle because of the presence of stagnation points. Minor errors on both the upstream and downstream slot lips due to the wedge approximation also can be seen.

Conclusions

1) An analytical design method for axially symmetrical ducts which allows direct prescription of desired pressure variation on the duct wall has been developed.

2) The method has been shown to be applicable to the design of both circular and annular cross-sectional ducts.

3) The method allows for the design of ducts with either prescribed velocity or prescribed flow direction on inlet and outlet equipotential lines.

4) A two-dimensional wedge flow approximation has been used successfully to circumvent the stagnation-point singularity encountered in the design of branched ducts.

5) The wedge flow approximation was used successfully in the design of both a circular and an annular cross-sectional diffuser with suction slots.

6) Favorable comparisons were obtained in all cases between the wall velocity variations prescribed to the design computer programs and wall velocity variations obtained by analyzing the designed geometries using the Douglas-Neumann method.

Acknowledgment

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